Alexandrov spaces with maximal number extremal points

Nina Lebedeva

December 5, 2011

Abstract

We show that n-dimensional nonnegatively curved Alexandrov space with maximal possible number of extremal points is isometric to a factor of $\mathbb{R}^n$ by a crystallographic group action and describe the actions which appear this way.

1 Introduction

A point p in Alexandrov space is called *extremal* if its space of directions have diameter $\leqslant \frac{\pi}{2}$. Equivalently, the one-point set $\{p\}$ is an *extremal set* as it defined by Perelman and Petrunin in [11]. Yet equivalently, p is a *critical point* of any distant function.

It was proved by Perelman that n-dimensional Alexandrov space with nonnegative curvature has at most 2^n extremal point. For completeness, we present this proof in Subsection 1A. This proof is a slight modification of the proof of the following problem in discrete geometry:

1.1. Problem. *Assume $x_1, x_2, \dots, x_m$ be a collection of points in the n-dimensional Euclidean space such that $\angle x_i x_j x_k \leqslant \frac{\pi}{2}$ for any distinct i, j and k. Show that $m \leqslant 2^n$ and moreover, if $m = 2^n$ then all x_i form set of vertexes of a right parallelepiped.*

This problem was posted Erdős in [6] and solved b Danzer and Grünbaum in [5].

In this paper we study nonnegatively curved n-dimensional Alexandrov spaces which have 2^n extremal points; further such space will be called *n-box*.

The question of classifying n-boxes can be characterized as folklore. Clearly, right parallelepipeds are boxes. In a private conversation around 1993, G. Perelman suggested that it might be the only examples. Soon it was noticed that this condition also holds for surface tetrahedra glued out of 4 equal triangles. Latter, it was suggested that all boxes have to be isometric to the factor of a flat torus by a group of isometries isomorphic to a product of $\mathbb{Z}_2$-groups. The later also turned out to be wrong — the first counterexample appears in dimension 3, this is the space $\square'_2$ constructed below.

Here is our main result:

[1]Partially supported by RFBR grant 11-01-00302a

1.2. Main theorem I. *Any n-box is isometric to the factor-space of $\mathbb{R}^n/\Gamma$ for a discrete cocompact isometric action $\Gamma \curvearrowright \mathbb{R}^n$.*

Main theorem II below describes the actions on $\mathbb{R}^n$ which can appear in Main theorem I. The following proposition implies that it is sufficient to describe the actions $\Gamma \curvearrowright \mathbb{R}^n$ up to affine conjugation.

1.3. Proposition. *Given a discrete action $\Gamma \curvearrowright \mathbb{R}^n$ by affine transformations and an invariant parallel metric g on $\mathbb{R}^n$, the factor $\mathcal{A}_g = (\mathbb{R}^n, g)/\Gamma$ is an Alexandrov space and a point $e \in \mathcal{A}_g$ is extremal if and only if it is an image of an isolated fixed point in $\mathbb{R}^n$ of some subgroup of Γ.*

In particular, the number of extremal points in $\mathcal{A}_g$ does not depend on the choice of g.

Recall that Coxeter group of an n-polyhedra is the group generated by reflections in all its faces; such group comes with an action on $\mathbb{R}^n$. Let us denote by $\Delta^n \curvearrowright \mathbb{R}^n$ the Coxeter action of unit cube.

1.4. Main theorem II. *Let $\Gamma \curvearrowright \mathbb{R}^n$ be a subaction of $\Delta^n \curvearrowright \mathbb{R}^n$ such that the stabilizer of any vertex e of the unit cube is an isolated fixed point for some subgroup of Γ. Then $\mathbb{R}^n/\Gamma$ is an n-box.*

Moreover an action may appear in the Main theorem I if and only if it is affine conjugate to an action described above.

It is immediate from theorem that $[\Delta^n : \Gamma] = 2^k$ for some $k \in \{0, \dots, n-1\}$.

Note that the above theorem makes possible to list all such group actions. Let S be a set of faces of the n-cube Q such that for any vertex e of Q we have

$$\bigcap_{e \in f \in S} f = \{e\}.$$

Then the group generated by reflections in the faces of S gives the example of our action and any our action can be obtained this way.

One can also glue any space $\mathcal{A}$ from Main theorem I from 2^k copies of the cube equipped with a parallel metric g which is invariant with respect to all reflections in faces of S.

Let us use this construction to classify the spaces in lower dimensions.

- $n = 1$ one space $\mathbb{I} = [0, 1]$; it has one parameter family of metrics.
- $n = 2$ two spaces: square $\square = \mathbb{I} \times \mathbb{I}$ and its doubling in the boundary $\square_2$. The square $\square$ admits two parameter family of metrics which, that are all possible rectangles. The doubled square $\square_2$ admits 3-parameter family of metrics, all of them are isometric to the surfaces of 3-simplexes with equal opposite sides; such simplexes are sometimes called *disphenoids*.
- $n = 3$, we have cube $\Cube = \mathbb{I} \times \mathbb{I} \times \mathbb{I}$; doubling of cube in the whole boundary $\Cube_2$; doubling of cube in the 5 faces $\Cube'_2$; the product $\Cube''_2 = \mathbb{I} \times \square_2$ and the factor of the standard torus by the central symmetry $\Cube_4$. The dimensions of the space of metrics are correspondingly 3, 3, 3, 4 and 6.
- one can continue, but we don't.

Comments and open questions. Given a discrete action by affine transformations $\Gamma \curvearrowright \mathbb{R}^n$, denote by $N(\Gamma)$ the number of orbits of isolated fixed point of some subgroups in Γ.

Further, denote by $M(\Gamma)$ the number of maximal finite subgroups in Γ up to conjugation. Note that if z is an isolated fixed point of a subgroup of Γ then

stabilizer of z is a maximal finite subgroup of Γ. It follows that $N(\Gamma) \leqslant M(\Gamma)$. Some maximal subgroups of Γ might fix affine subspaces of positive dimension, therefore $M(\Gamma)$ might be strictly more that $N(\Gamma)$. From Proposition 1.3 and Main theorem I (1.2), we have the following:

1.5. Corollary. *For any cocompact discrete action by affine transformations. $\Gamma \curvearrowright \mathbb{R}^n$, we have $N(\Gamma) \leqslant 2^n$.*

We believe that the following conjecture is true.

1.6. Conjecture. *For any cocompact discrete action by affine transformations $\Gamma \curvearrowright \mathbb{R}^n$, we have $M(\Gamma) \leqslant 2^n$.*

A big part of my proof works for this conjecture, but there is a apparently small gap which I can not pass, see my question on mathoverflow [9].

Structure of the proof. Fist we show that $\mathcal{A}$ has to be a polyhedral space (Theorem 3.1). According to Proposition 2.12, it is sufficient to show that each point $p \in \mathcal{A}$ has a conic neighborhood (see Definition 2.5). This is proved in Key lemma 3.2.

Further, we show that angle around any face of codimension 2 in $\mathcal{A}$ has to be π, or 2π. By Proposition 2.14, it implies the Main theorem I (1.2).

To prove Main theorem II, it is sufficient to show that a set of points fixed by some subgroup forms a lattice. We find two technical properties of group that are used to prove this and show that a group action for a box possesses this properties.

The author thanks Anton Petrunin for bringing this problem to my attention and for useful discussions.

1A Proof that upper bound is 2^n.

In this subsection we give Erdős–Danzer–Grünbaun–Perelman's proof of the following theorem; we also introduce notations which will be used further.

1.7. Theorem. *The number of extremal points of an n-dimensional nonnegatively curved Alexandrov space is at most 2^n.*

Proof. Let $\mathcal{A}$ be an n-dimensional nonnegatively curved Alexandrov space. Label the extramal points in $\mathcal{A}$ as $e_1, e_2, \ldots, e_m$.

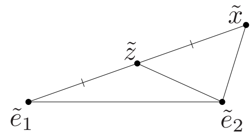
1.8. Lemma. *Let $\mathcal{A}$ be Alexandrov space with curvature $\geqslant 0$ and $e_1, e_2 \in \mathcal{A}$ be two extremal points. Assume z is a midpoint of a shortest path $[e_1x]$ in $\mathcal{A}$, then*

$$|e_1z| \leqslant |e_2z|.$$

More over, if $|e_1z| = |e_2z|$ then

$$\angle e_1ze_2 = \tilde{\angle} e_1ze_2, \quad \angle e_2zx = \tilde{\angle} e_2zx$$

and there is a unique flat subgeodesic triangle e_1e_2x with given median $[e_2z]$.



Proof of the claim. Suppose contrary; i.e $|e_1z| > |e_2z|$ for some j. Consider model triangle $\tilde{e}_1\tilde{e}_2\tilde{z}$ for e_1e_2z in the plane. Let $\tilde{x}$ be a point on the line extension of $\tilde{e}_1\tilde{z}$ such that $|\tilde{x}\tilde{z}| = |\tilde{z}\tilde{e}_1|$. Since $|\tilde{e}_1\tilde{z}| > |\tilde{e}_2\tilde{z}|$, we have $\angle\tilde{e}_1\tilde{e}_2\tilde{x} > \pi/2$. From triangle comparison, we have $|e_2x| \leqslant |\tilde{e}_2\tilde{x}|$. It follows that

$$\angle e_1e_2x \geqslant \tilde{\angle} e_1e_2x \geqslant \angle\tilde{e}_1\tilde{e}_2\tilde{x} > \pi/2.$$

In particular, $\operatorname{diam}\Sigma_{e_2} > \pi/2$, a contradiction.

In the case of equality $|e_1z| = |e_2z|$ we use the same comparison picture as above, then we have $\angle\tilde{e}_1\tilde{e}_2\tilde{x} = \pi/2$. Suppose contrary $\angle e_1ze_2 > \tilde{\angle} e_1ze_2$, then from triangle comparison we obtain $|e_2x| < |\tilde{e}_2\tilde{x}|$ and hence

$$\angle e_1e_2x \geqslant \tilde{\angle} e_1e_2x > \angle\tilde{e}_1\tilde{e}_2\tilde{x} = \pi/2$$

and we obtain a contradiction, proving angle equalities. Now the existence of flat subgeodesic triangle follows from Lemma 2.1.

$\square$

Now, let us introduce two additional notations:

1. Set W_i to be the set of midpoints of all geodesics $[e_ix]$ with $x \in \mathcal{A}$.
2. Set V_i to be the Voronoi domain of e_i; i.e.

$$V_i = \{\, x \in \mathcal{A} \mid |e_ix| \leqslant |e_jx| \text{ for any } i \,\}$$

From Lemma 1.8, we have $W_i \subset V_i$ for all i.

Further, consider map $\varphi_i\colon W_i \to \mathcal{A}$, defined the following way: $x = \varphi(z)$ if z is a midpoint of a geodesic $[e_ix]$.

By triangle comparison, we have

$$|\varphi_i(z)\ \varphi_i(z')| \leqslant 2{\cdot}|zz'|$$

for any $z, z' \in W_i$. In particular, the map φ_i is uniquely defined.

Hence

$$\operatorname{vol} V_i \geqslant \operatorname{vol} W_i \geqslant \tfrac{1}{2^n}\cdot\operatorname{vol}\mathcal{A}.$$

Since

$$\sum_{i=1}^{m} \operatorname{vol} V_i = \operatorname{vol}\mathcal{A},$$

we get $m \leqslant 2^n$. $\square$

Note that from the proof we immediately get the following:

1.9. Corollary. *Let $\mathcal{A}$, n, m, V_i and W_i be as in the proof of theorem 1.7. Assume $m = 2^n$ then $W_i = V_i$ and $\operatorname{vol} V_i = \frac{1}{2^n}\cdot\operatorname{vol}\mathcal{A}$ for all i.*

2 Preliminary statements.

In this section we prove number of technical statements needed in the main proof. Further we denote by $\mathcal{A}$ an Alexandrov space.

2A Flat slices in Alexandrov space.

2.1. Lemma. *Let $\mathcal{A}$ be n-dimensional Alexandrov space with nonnegative curvature and $[px_1], [px_2], \dots, [px_k]$ be geodesics in $\mathcal{A}$.*

Assume that

$$\angle x_i p x_j = \tilde{\angle} x_i p x_j$$

for all i, j and all directions $\uparrow_{[px_i]}$ lie in a subcone, say E of $T_p\mathcal{A}$ which is isometric to a convex cone in Euclidean space.

Then all geodesics $[px_i]$ lie in a subset of $\mathcal{A}$ which is isometric to a convex polyhedra in Euclidean space.

Proof. Set $\tilde{x}_i = \log_p x_i \in T_p$ and $\tilde{p} = \log_p p$ ($\tilde{p}$ is the vertex of T_p). Clearly

- $\diamond$ $\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_k \in E$.
- $\diamond$ $|px_i| = |\tilde{p}\tilde{x}_i|$ for each i;
- $\diamond$ $|x_i x_j| = |\tilde{x}_i \tilde{x}_j|$ for all i, j

Since E is Euclidean, by Kirszbraun theorem, there is a short map $s : \mathcal{A} \to E$ such that $s(p) = \tilde{p}$ and $s(x_i) = \tilde{x}_i$ for each i.

On the other hand the gradient exponent gexp_p is also short map. Thus the composition $f = s \circ \mathrm{gexp}_p$ is also short. Clearly f does not move $\tilde{x}_i$ and $\tilde{p}$. It follows that f does not move any point in $Q = \mathrm{Conv}(\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_k)$. Therefore, gexp_p maps Q isometrically in $\mathcal{A}$. $\square$

2B Cones, splitting and affine functions.

2.2. Definition. *Let $\Omega \subset \mathcal{A}$ be an open subset and $\lambda \in \mathbb{R}$. A locally Lipschitz function $f\colon \Omega \to \mathbb{R}$ is called λ-affine if*

$$(f \circ \gamma)''(t) \equiv \lambda$$

for any unitspeed geodesic γ in Ω. 0-affine functions will be also called affine.

For Alexandrov space $\mathcal{A}$ its subset $\Omega \subset \mathcal{A}$ and a function $f\colon \Omega \to \mathbb{R}$ we will denote by $\overline{\mathcal{A}}$ the doubling of $\mathcal{A}$, by $\overline{\Omega} \subset \overline{\mathcal{A}}$ the doubling of Ω and by $\overline{f} : \overline{\Omega} \to \mathbb{R}$ the tautological extension of f.

2.3. Definition. *We say that λ-affine $f\colon \Omega \to \mathbb{R}$ satisfies boundary condition if $\overline{f} : \overline{\Omega} \to \mathbb{R}$ is λ-affine.*

For $i \in \{1, 2\}$, assume $f_i\colon \Omega \to \mathbb{R}$ be λ_i-affine function. Clearly, $f_1 + f_2$ is a $(\lambda_1 + \lambda_2)$-affine. Moreover for any real constant c, the function $c{\cdot}f_1$ is $(c{\cdot}\lambda_1)$-affine.

For the proof of Proposition 2.4, Proposition 2.6 and Claim 2.7 we refer the paper [1], functions regarded in this paper are defined on the whole Alexandrov space, but the proof works successfully also for our local case, we just have to note that every shortest path between points in $B_{r/4}(p)$ is inside $B_r(p)$.

2.4. Proposition. *Let $f_1, f_2, \dots, f_k$ be affine functions defined in a ball $B_r(p)$ such that the functions $1, f_1, f_2 \dots, f_k$ form a linear independent system. Then the ball $B_{r/4}(p)$ is isometric to an open set in a product $\mathbb{R}^k \times X$ for some metric space X.*

2.5. Definition. *A point $p \in \mathcal{A}$ admits a* conic neighborhood *if there is an isometry from a neighborhood of p to an open set in a Euclidean cone which sends p to the vertex of the cone.*

2.6. Proposition. *Suppose a ball $B_r(p)$ admits an 1-affine function f. Then the ball $B_{r/4}$ is isometric to an open set in a Euclidean cone. In particular case $\nabla_p f = 0$ we have $f = \frac{1}{2}\operatorname{dist}_p^2 + c$ and the ball $B_{r/4}(p)$ is a cone neighborhood of p.*

2.7. Claim. *Let f be λ-affine function defined in some neighborhood $U \ni p$ and satisfies boundary condition. Then for gradient $\nabla_p f$ there is an opposite vector $-\nabla_p f \in T_p A$ and $d_p f = \langle \nabla_p f, \cdot \rangle$.*

For a set F of affine functions defined in some neighborhood $U \ni p$ we denote by $\#_L(F, p)$ the maximal number of functions in F, say $f_1, \dots, f_k$, so that the functions $1, f_1, f_2 \dots, f_k$ form a linear independent system in some small ball $B_r(p) \subset U$. We note that since affine function on every geodesic is determined by initial value and initial derivative this number does depend on the choice of such a ball.

For a set F of 1-affine functions defined in some neighborhood $U \ni p$ we define a set of affine functions $F^0 = \{\sum \alpha_i f_i | f_i \in F, \alpha_i \in \mathbb{R}, \sum \alpha_i = 0\}$ and denote by $\#_A(F, p) = \#_L(F^0, p)$.

It follows from Claim 2.7 that gradients of functions in F are in linear subspace of $T_p A$, hence we can define numbers: $\#_L(\nabla F, p)$ – the dimension of vector subspace in $T_p \mathcal{A}$, generated by gradients of functions in F and $\#_A(\nabla F, p)$ – the dimension of affine subspace, generated by endpoints of this gradients.

2.8. Claim. *Let F be a finite set of affine (1-affine) functions defined in a ball $B_r(p)$.*

Then for affine function we have: $\#_L(F, p) = \#_L(\nabla F, p)$, and $\#_A(F, p) = \#_A(\nabla F, p)$ for 1-affine.

Proof It follows from Claim 2.7 that differential of every affine (1-affine) function is uniquely defined by its gradient and hence every affine (1-affine) function $f : B_r(p) \to \mathbb{R}$ is defined by $f(p)$ and $\nabla_p f$. Now the proof is straightforward. □

2.9. Corollary. *Let F be a finite set of 1-affine functions defined in a ball $B_r(p)$.*

There are two possibilities:

(1) if $\#_L(\nabla F, p) = \#_A(\nabla F, p) + 1$, then the ball $B_{r/4}(p)$ is isometric to an open subset of $\mathbb{R}^{\#_A(\nabla F, p)} \times Cone$.

(2) if $\#_L(\nabla F, p) = \#_A(\nabla F, p)$, then the ball $B_{r/4}(p)$ is a conic neighborhood of p isometric to subset of $\mathbb{R}^{\#_A(\nabla F, p)} \times Cone$.

proof Let regard the set $F^0 = \{\sum \alpha_i f_i | f_i \in F, \alpha_i \in \mathbb{R}, \sum \alpha_i = 0\}$. Then F^0 is a set of affine functions and $\#_L(\nabla F_0, p) = \#_A(\nabla F, p)$, hence by Claim 2.8 and Proposition 2.4 we obtain $\mathbb{R}^{\#_A(\nabla F, p)}$ factor in decomposition for both (1) and (2).

In the case (2) in addition there are numbers α_i, so that $\sum \alpha_i = 1$ and $\sum \alpha_i \nabla_p f_i = 0$. Then the function $f = \sum \alpha_i f_i$ is 1-affine and $\nabla_p f = 0$. Hence

by Proposition 2.6 $f = \frac{1}{2}\operatorname{dist}_p^2 + c$ and the ball $B_{r/4}(p)$ is a cone neighborhood of p. ☐

Given a semiconcave function $f\colon \mathcal{A} \to \mathbb{R}$, we will denote by $\Phi_f^t\colon \mathcal{A} \to \mathcal{A}$ the corresponding gradient flow for time t.

2.10. Theorem. *Let f be a λ-concave function and $\Omega \subset \mathcal{A}$ be an open set. Then for any $t > 0$, we have*

$$\operatorname{vol} \Phi_f^t(\Omega) \leqslant \exp(n\cdot\lambda\cdot t)\cdot \operatorname{vol}\Omega.$$

Moreover if equality holds for some $t > 0$, then f is λ-affine in Ω and satisfies boundary condition

Proof. Here we denote γ'_- denotes velocity of a curve γ if we go backwards.

λ-concavity of function f means that

$$d_p f(\gamma'(a)) + d_q f(\gamma'_-(b)) \geqslant -\lambda|pq|$$

for any unit speed shortest path γ in Ω between p and q; to prove that f is λ-affine it is sufficient to show that this inequality became equality. We regard gradient curves $p(t)$ and $q(t)$ and let l be the distance function $l(t) = |p(t)q(t)|$.

By first variation formula

$$l'(t) \leqslant -(\langle\gamma'(a), \nabla_p f\rangle + \langle\gamma'_-(b), \nabla_q f\rangle)$$

by definition of gradient for every point x and $v \in T_x\mathcal{A}$ we have $\langle v, \nabla_x f\rangle \geqslant d_x f(w)$. So we have:

$$l'(t) \leqslant \lambda|pq|$$

and applying Proposition 2.16 we obtain the required volume inequality and in the case this inequality became equality we have that $l'(t) = \lambda|pq|$, hence

$$d_p f(\gamma'(a)) = \langle\gamma'(a), \nabla_p f\rangle, \quad d_q f(\gamma'_-(b)) = \langle\gamma'_-(b), \nabla_q f\rangle$$

and λ-concavity follows.

To prove the boundary condition it is sufficient to check 1-affinity on every shortest path $\gamma : [-h, h] \to \overline{\Omega}$ intersecting $\overline{\partial\mathcal{A}}$ only once at point $x = \gamma(0) \in \overline{\partial\mathcal{A}}$. Clearly, it is sufficient to prove that $d_x\overline{f}(-\gamma'(0)) = -d_x\overline{f}(\gamma'(0))$.

By above for every $x \in \mathcal{A}\cap\Omega$ the differential $d_x f = \langle\nabla_x f, \cdot\rangle$ and hence in particular the gradient has an opposite vector in tangent space. Then for every $x \in \partial\mathcal{A}\cap\Omega$ both vectors $\nabla_x f, -\nabla_x f \in \partial T_x\mathcal{A}$ and are glued with themselves under doubling. Hence for the doubling function we have: the gradient $\nabla\overline{f}$ has an opposite vector and $d_x\overline{f} = \langle\nabla_x\overline{f}, \cdot\rangle$. Then $\angle(-\gamma'(0), \nabla f) = \pi - \angle(\gamma'(0), \nabla f)$ and $d_x\overline{f}(\gamma'_-(0)) = -d_x\overline{f}(\gamma'(0))$.

☐

2C Polyhedral spaces.

2.11. Definition. *A metric on a simplicial complex $\mathcal{S}$ is called* polyhedral *if each simplex in $\mathcal{S}$ is isometric to a simplex in a Euclidean space.*

A metric space $\mathcal{P}$ is called polyhedral space *if it is isometric to a simplicial complex with a polyhedral metric.*

The following characterization of polyhedral space seems to be classical, but I was not be able to find it in the literature.

2.12. Proposition. *Let X be a compact length space. Assume that each point $x \in X$ has a conic neighborhood. Then X is polyhedral space.*

In the proof we mimic construction of Delone triangulation for X.

Proof. Consider a finite cover of X by open balls $B(x_i, r_i)$ such that for each i the ball $B(x_i, 7{\cdot}r_i)$ forms a cone neighborhood of x_i. For each i consider function $f_i(z) = |x_i z|^2 - r_i^2$. We will call $f(z)$ *power of z with respect to sphere of radius r_i centered* at x_i. Clearly $f_i(z) < 0$ iff $z \in B(x_i, r_i)$. Consider corresponding Voronoi domain

$$V_i = \{\, z \in X \mid f_i(z) \leqslant f_j(z) \text{ for any } i \,\}.$$

Set $f = \min_i f_i$, it is a continuous function on X. Note that $f < 0$, in particular $V_i \subset B(x_i, r_i)$.

We may assume that radii are chosen generically; i.e. if $z \in \cap_{i \in Q} V_i$ for some index set Q then the functions $\{\, f_i \mid i \in Q \,\}$ are linearly independent in arbitrary neighborhood of z.

Consider nerve of covering $\{V_i\}$ of X; it is an abstract simplicial complex $\mathcal{S}$ with set of vertexes in the index set of x_i and an index subset Q forms a simplex $\bigcap_{i \in Q} V_i \neq \emptyset$.

Notice that vertexes of any simplex $\triangle^k$ in $\mathcal{S}$ can be reordered as $i_0, i_1, \dots, i_k$ on such a way that $r_{i_0} \leqslant r_{i_1} \leqslant \dots \leqslant r_{i_k}$. Clearly $x_{i_n} \in B(x_{i_m}, 3{\cdot}r_{i_m})$ for all $n \leqslant m$. Let us construct a map $\triangle^k \to X$.

1. map $i_0 \mapsto x_{i_0}$;
2. map $i_1 \mapsto x_{i_1}$ and use cone structure in $B(x_{i_1}, 3{\cdot}r_{i_1})$ to extend it linearly to 1-simplex $i_0 i_1$;
3. map $i_2 \mapsto x_{i_2}$ and use cone structure in $B(x_{i_2}, 3{\cdot}r_{i_2})$ to extend it linearly to 2-simplex $i_0 i_1 i_2$;
4. and so on.

It is straightforward to check that simplex with metric induced by this map is isometric to a simplex in Euclidean space. Further, this map agree on intersections of different simplexes of $\mathcal{S}$, hence we get a map $\imath \colon \mathcal{S} \to X$.

It only remains to show that $\imath(\mathcal{S}) = X$. Assume contrary; i.e., $\Omega = X \backslash \imath(\mathcal{S}) \neq \emptyset$. For each $x \in X$ choose a closest point $x^* \in \imath(\mathcal{S})$. Note that

$$f(x) > f(x^*)$$

for all $x \in \Omega$ sufficiently close to $\imath(\mathcal{S})$. It follows that there is a point $x_0 \in \Omega$, of local maximum of f. Let Q be the a subset of the index set, such that $i \in Q$ iff $V_i \ni x_0$; denote by Δ the simplex corresponding to Q. Since x_0 is a maximum point of f, we get $x_0 \in \Delta$, a contradiction. □

2C.1 One fact about polyhedral space cutlocus.

We denote cells in Voronoy decomposition of a metric space X with respect of points $x_1, \dots, x_l \in X$ correspondently: $\mathfrak{V}_{x_k}(X, x_1, \dots, x_l)$, where $k = 1, \dots, l$.

2.13. Lemma. *Let P be a polyhedral space, a point $e \in P$, C be cutlocus for e and K completion of $P \setminus C$ and g correspondent gluing map $g : K \to$*

P. Let for $x \in C$ there is l shortest paths between x and e: $s_1, \dots, s_l$ or equivalently l preimages: $g^{-1}(x) = \{x_1, \dots, x_l\} \subset \partial K$ (here s_k is the image under g of shortest path between x_k and e for $k = 1, \dots, l$). Let regard for points $s'_1(0), \dots, s'_l(0)$ in unit tangent space $\Sigma_x P$ correspondent Voronoy sets: $\mathfrak{V}_{s'_k(0)}(\Sigma_x, s'_1(0), \dots, s'_l(0))$ for $k = 1, \dots, l$. Then

$$d_{x_k} g(\Sigma_{x_k} K) = \mathfrak{V}_{s'_k(0)}(\Sigma_x, s'_1(0), \dots, s'_l(0))$$

for any $k \in \{1, \dots, l\}$.

Proof is left to the reader.

2D Orbifolds.

Here is characterizing property of flat orbifolds among polyhedral spaces.

2.14. Proposition. *A polyhedral space $P = (\mathcal{S}, d)$ is isometric to a factor space $\mathbb{R}^n/\Gamma$, for a discrete action by isometries $\Gamma \curvearrowright \mathbb{R}^n$ if and only if*

1. *A simplicial complex $\mathcal{S}$ of P is an n-dimensional pseudomanifold; i.e. $\mathcal{S}$ is connected; any simplex in $\mathcal{S}$ is a face of a simplex of dimension n; the link of any simplex of dimension $\leqslant n-2$ is connected; any simplex of dimension $n-1$ belongs to at most two simplexes of dimension n.*
2. *For any point x on a face F of codimension 2 in P, the normal cone $N_x F$ of F at x is isometric to a factor of $\mathbb{R}^2$ by a subgroup of rotations. Namely, $N_x F$ has to be isometric to a cone over S^1 with length $2 \cdot \pi/k$ or to a cone over interval of length π/k for some $k \in \mathbb{N}$.*

Proof.

We will use the fact that for any orbifold admitting constant curvature the universal branched cover is a manifold.

So it is sufficient to check that P is an orbifold, i.e. for any point x in P tangent space is of the form $\mathbb{R}^n/\Gamma$.

Actually it is convenient to prove simultaneously the same statement as in our Proposition but changing polyhedral space by spherical polyhedral space and correspondently $\mathbb{R}^n/\Gamma$ by S^n/Γ. So let call 'good' space polyhedral or spherical polyhedral if it possesses 1. 2.

We prove by inverse induction on dimension that 'good' space is $\mathbb{R}^n/\Gamma$ or S^n/Γ.

The base $n = 2$ follows because of condition 2. Suppose 'good' space of dimension $n-1$ is $\mathbb{R}^n/\Gamma$ or S^n/Γ. Then for any n-dimensional 'good' space P and any point $x \in P$ the unit tangent space $U_x P$ is a spherical polyhedral space that inherited properties 1. and 2. and hence is 'good'. Hence by induction hypothesis $U_x P = S^n/\Gamma$ and P is an orbifold, this proves induction step.

$\square$

Here is one auxiliary fact about spherical orbifold, the proof is straightforward and is left to the reader.

2.15. Lemma. *Let S^k be a k-dimensional sphere, Γ discrete subgroup of isometries of S^k, $B = S^k/\Gamma$ with projection $p : S^k \to B$ and diam$B < \pi$. Let point $x \in B$, $\mathfrak{C}_x$ denote cutlocus of x and $p^{-1}(x) = \{x_1, \dots, x_m\}$. For Voronoy decomposition of S^k with respect to the set of points $\{x_1, \dots, x_m\}$ we have the following:*

(1) $y \in \mathfrak{V}_{x_i}(S^k, \{x_1, \dots, x_m\})$ *iff* $|x_i y| = |xp(y)|$.
(2) $y \in \partial\mathfrak{V}_{x_i}(S^k, \{x_1, \dots, x_m\})$ *iff* $p(y) \in \mathfrak{C}_x$.

2E Volume preserving + 1-Lipschitz = isometry

2.16. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be m-dimensional Alexandrov spaces, $\Omega \subset \mathcal{X}\backslash\partial\mathcal{X}$ be an open set and $f\colon \Omega \to Y$ be a 1-Lipschitz volume preserving map. Then f is a locally distance preserving; i.e., for any point $x \in \Omega$ there is a neighborhood $\Omega_x \ni x$ such that the restriction $f|\Omega_x$ is a distance preserving map.*

Rem?rk. As far as we know no one bothered to write a proof of the above proposition. It looks a bit strange since it is useful in many problems. For example, one can use it to prove the equality case in the Bishop–Gromov inequality.

2.17. Equality case in Bishop–Gromov inequality. *Assume $\mathcal{X}$ be an m-dimensional Alexandrov space with curvature $\geqslant \kappa$ and for some point $p \in \mathcal{X}$ the volume of $B_R(p)$ coincide with the volume of R-ball in the model space $\mathbb{M}^m[\kappa]$ then $B_R(p)$ is locally isometric to the R-ball in the model space $\mathbb{M}^m[\kappa]$.*

In the proof we will need the following statements.

2.18. Domain invariance theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be m-dimensional Alexandrov spaces, and $\Omega \subset \mathcal{A}\backslash\partial\mathcal{A}$ be an open set. Then any continuous injective map $f\colon \Omega \to \mathcal{B}$ is open.*

Proof. We will need the following lemma proved by Grove and Petersen; see [7, Lemma 1].

2.19. Lemma. *Let $\mathcal{A}$ be a compact m-dimensional Alexandrov space without boundary. Then $\mathcal{A}$ has a fundamental class in Alexander–Spanier cohomology with $\mathbb{Z}_2$ coefficients; i.e., $\bar{H}^m(\mathcal{A}, \mathbb{Z}_2) = \mathbb{Z}_2$.*

Note that Domain invariance theorem follows directly from this lemma if $\mathcal{A}$ is compact and $\Omega = \mathcal{A}$. Moreover, this argument also implies the following:

2.20. Claim. *Let $\mathcal{A}$ and $\mathcal{B}$ be compact m-dimensional Alexandrov spaces without boundary. Assume $f\colon \mathcal{A} \to \mathcal{B}$ be a continuous map which is injective on some open subset; i.e., for some open set $U \subset \mathcal{A}$, we have $x \in U$ and $f(x) = f(y)$ implies $x = y$. Then f is surjective.*

Now let us do the general case. Let $y = f(x)$ for some $x \in \Omega$. Let us use f to construct a continuous map between spherical suspensions over spaces of directions $\mathbb{S}(\Sigma_x) \to \mathbb{S}(\Sigma_y)$ which is injective around one point in the target. To do this, take a small spherical neighborhood $W \ni y$. According to Perelman's theorem [10], W is homeomorphic to $\operatorname{Cone}\Sigma_x$. In other words, there is an embedding of $W \hookrightarrow \mathbb{S}(\Sigma_y)$ which image is whole $\mathbb{S}(\Sigma_y)$ without south pole. Collapsing everything outside of W to the south pole, we get a continuous map $\mathcal{B} \to \mathbb{S}(\Sigma_y)$ which is injective everywhere in W.

Without loss of generality, we may assume that $f^{-1}(W)$ lies in a spherical neighborhood of x which is homeomorphic to $\mathbb{S}(\Sigma_x)$ without south pole. The

composition of constructed maps, give a map $\mathbb{S}(\Sigma_x) \to \mathbb{S}(\Sigma_y)$ which satisfies the conditions of the claim; thus it is surjective and hence f is an open map. $\square$

2.21. Lemma. *Let $\mathcal{X}$ and $\mathcal{Y}$ be m-dimensional Alexandrov spaces, $\Omega \subset \mathcal{X}$ be an open set and $f\colon \Omega \to \mathcal{Y}$ be a locally Lipschitz map. Assume $d_x f$ is 1-Lipschitz for almost all $x \in \Omega$ then f is locally 1-Lipschitz.*

Proof. Given $x \in \Omega$, choose $\varepsilon > 0$, such that $B_{2\cdot\varepsilon}(x) \subset \Omega$ and f is Lipschitz in $B_{2\cdot\varepsilon}(x)$ with constant say L.

Choose arbitrary points $v, w \in B_\varepsilon(x)$. Fix small positive $r < \varepsilon - |xw|$. Note that any minimizing geodesic $[vz]$ which $z \in B_r(w)$ lies in $B_{2\cdot\varepsilon}(x)$. Denote by $\gamma_z\colon [0,1] \to \mathcal{X}$ the constant speed parametrization of $[vz]$. From Toponogov comparison theorem, for any $t \in (0,1]$, the set

$$Z_t = \{\ \gamma_z(t) \mid z \in B_r(w)\ \}$$

has positive volume. In particular, differential $d_{\gamma_z(t)} f$ is 1-Lipschitz for almost all $t \in [0,1]$ and $z \in B_r(w)$. Hence

$$\begin{aligned}\operatorname{length}(f \circ \gamma_z) &= \int_0^1 d_{\gamma_z(t)} f(\gamma_z^+(t))\cdot dt \leqslant \\ &\leqslant \int_0^1 |\gamma_z^+(t)|\cdot dt \leqslant \\ &\leqslant \operatorname{length}\gamma_z\end{aligned}$$

for almost all $z \in B_r(w)$. It follows that

$$|f(v)f(w)| \leqslant |vw| + (1+L)\cdot r.$$

Since r can be chosen arbitrary small, the lemma follows. $\square$

Proof of Proposition 2.16. Set

$$\mathfrak{C} = \left\{\ x \in \Omega \,\middle|\, \operatorname{vol}^{m-1}\Sigma_x \leqslant \tfrac{1}{2}\cdot \operatorname{vol}^{m-1}\mathbb{S}^{m-1}\ \right\} \qquad \text{and} \qquad \Omega' = \Omega \backslash \mathfrak{C}.$$

$\mathfrak{C}$ is a closed nowhere dense subset relatively Ω, since $\partial\mathcal{X} \cap \Omega = \emptyset$, we have that $\dim_H \mathfrak{C} < m-1$ and hence $\dim_H f(\mathfrak{C}) < m-1$, where $\dim_H$ denotes Hausdorff dimension.

From [13], we get that Ω' is locally convex in Ω. Moreover, for any point $x \in \Omega$ there is a neighborhood $\Omega_x \ni x$, such that if $[vw]$ a minimizing geodesic with $v \in \Omega' \cap \Omega_x$ and $w \in \Omega_x$ then $[vw[\subset \Omega'$; here $[vw[= [vw]\backslash\{w\}$.

Let us show that the restriction $f|\Omega'$ is injective. Assume contrary; i.e., $p = f(x) = f(y)$ for some $x, y \in \Omega'$. From Bishop–Gromov inequality, there is $\varepsilon > 0$ such that for all $r < \varepsilon$ we have

$$\operatorname{vol}^m B_r(x),\ \operatorname{vol}^m B_r(y) > \tfrac{1+\varepsilon}{2}\cdot r^m\cdot\beta_m,$$

where β_m denotes the volume of unit ball in Euclidean space. Therefore

$$\operatorname{vol}^m B_r(p) > (1+\varepsilon)\cdot r^m\cdot\beta_m$$

for all sufficiently small r. The later contradicts Bishop–Gromov inequality for $B_r(p)$.

Now we will use the same idea to prove that the restriction $f|\Omega'$ is locally bi-Lipschtz. Assume contrary; i.e., there is $x_\infty \in \Omega'$ and two sequences of points $x_n, x'_n \to x_\infty$ such that

$$\frac{|f(x_n) - f(x'_n)|}{|x_n - x'_n|} \to 0. \qquad ❶$$

Set $r_n = \frac{|x_n - x'_n|}{2}$ and $s_n = |f(x_n) - f(x'_n)|$. Clearly

$$\begin{gathered} f(B_{r_n}(x_n)),\ f(B_{r_n}(x'_n)) \subset B_{r_n+s_n}(f(x_n)), \\ B_{r_n}(x_n) \cap B_{r_n}(x'_n) = \emptyset. \end{gathered} \qquad ❷$$

Note that from Bishop–Gromov inequality, there is $\varepsilon = \varepsilon(x_\infty) > 0$ such that the following two inequalities

$$\begin{gathered} \operatorname{vol}^m B_{r_n}(x_n),\ \operatorname{vol}^m B_{r_n}(x'_n) > \tfrac{1+\varepsilon}{2}\cdot r_n^m \cdot \beta_m \\ \operatorname{vol}^m B_{r_n+s_n}(f(x_n)) < (1 + \tfrac{\varepsilon}{2})\cdot(r_n + s_n)^m \cdot \beta_m \end{gathered} \qquad ❸$$

hold for all sufficiently large n.

Since f is volume preserving, inequalities ❷ and ❸ imply that

$$(1 + \tfrac{\varepsilon}{2})\cdot(r_n + s_n)^m > (1 + \varepsilon)\cdot r_n^m$$

for all large n. The later contradicts ❶.

Applying domain invariance, we get the set $f(\Omega')$ is open. From above, the inverse of the restriction $h = (f|\Omega')$ is defined and locally Lischitz map with $f(\Omega')$ as the domain of definition.

From coarea formula, we obtain that $d_x f$ is isometry for almost all $x \in \Omega$. It follows that $d_y h = (d_{h(y)} f)^{-1}$ is also an isometry for almost all $y \in f(\Omega')$. Applying Lemma 2.21, we get that h is locally 1-Lipschitz; i.e., the restriction $f|\Omega'$ is locally distance preserving.

It remains to show that f is a distance preserving in a neighborhood of any given point $x \in \mathfrak{C}$. Let us choose $\varepsilon > 0$ so that $B_{2\cdot\varepsilon}(x) \subset \Omega$ and show that the restriction $f|B_\varepsilon(f(x))$ is a distance preserving. To do this we apply argument similar to the proof of Lemma 2.21 twice.

First let us show that $f(\Omega')$ is dense in $B_\varepsilon(f(x))$. Assume contrary, then choose $v \in f(\Omega' \cap B_\varepsilon(x))$ and w such that $B_r(w) \subset B_\varepsilon(f(x)) \backslash f(\Omega')$ for some $r > 0$.

Set $S_r = \{\, z \in B_r(w) \mid |vz| = |vw| \,\}$. Each geodesic $[vz]$ with $z \in S_r$ contains a point outside of $f(\Omega')$. Since $f(\Omega')$ is open, we can choose the the point $q_z \in [vz] \backslash f(\Omega')$ which is closest to v. Let $Q_r = \{\, q_z \mid z \in S_r \,\}$. By Toponogov comparison theorem, the map $Q_r \to S_r$, defined as $q_z \mapsto z$ is Lipschitz. Hence the $\dim_H Q_r \geqslant m - 1$.

Note that $Q_r \subset B_{2\cdot\varepsilon}(f(x))$, therefore $Q_r \subset f(\mathfrak{C})$. The later contradicts that $\dim_H f(\mathfrak{C}) < m - 1$.

It remains to show that for any $v, w \in B_\varepsilon(f(x)) \cap f(\Omega')$ we have

$$|h(v)(w)| \leqslant |vw|. \qquad ❹$$

Fix small positive $r > 0$, such that $B_r(w) \subset B_\varepsilon(f(x)) \cap f(\Omega')$. And set as above $S_r = \{\, z \in B_r(w) \mid |vz| = |vw| \,\}$. The same argument as above shows that for

almost all $z \in S_r$, a minimizing geodesic $[vz]$ lies in $f(\Omega')$. Since $h\colon f(\Omega') \to \mathcal{X}$ is locally distance preserving for any such z we get $|h(v)h(z)| \leqslant |vz|$. Hence $|h(v)h(w)| \leqslant |vw| + r$; since $r > 0$ can be chosen arbitrary small we get ❹. $\square$

3 Box is a polyhedral space.

In this section we prove the following main result:

3.1. Theorem. *Any box is a polyhedral space.*

Let $\mathcal{A}$ be an n-box. Let us keep the notations for e_i, V_i, W_i and φ_i for all $i \in \{1, 2, \ldots, 2^n\}$ as in Subsection 1A. According to corollary 1.9, $V_i = W_i$ for all i.

As it follows from Proposition 2.12, it is sufficient to prove the following lemma:

3.2. Key lemma. *Any point $x \in \mathcal{A}$ admits a cone neighborhood.*

Further let us denote by $\mathfrak{C}_i$ the cutlocus of e_i; i.e. the set of points in $z \in \mathcal{A}\backslash\{e_i\}$ which do not lie in the interior of some shortest path $[e_i x]$.

3.3. Proposition. *Each function $f_i = \frac{1}{2} \cdot \operatorname{dist}^2_{e_i}$ is 1-affine and satisfies boundary condition in $\mathcal{A} \setminus \mathfrak{C}_i$.*

Proof. It is sufficient to note that φ_i coincides with the restriction $\Phi^{\ln 2}_{f_i}|_{W_i}$ on whole domain of definition. Then from Corollary 1.9 and Theorem 2.10 follows that f_i is 1-affine and satisfies boundary condition.

$\square$

About the proof of Key Lemma. First, let us introduce necessary notations. Given a point $x \in \mathcal{A}$, consider index set J_x such that $i \in J_x$ iff $x \in V_i$.

From Proposition 3.3, it follows that if $x \in V_i$ then f_i is 1-affine in a neighborhood of x. Given a point $x \in \mathcal{A}$ set $\#(x) = \#_A\{f_i | i \in J_x\}$ (definition in section 2B).

According to 2.8 and 2.9, $\#(x) \leqslant n$ for any $x \in \mathcal{A}$. Moreover if $\#(x) = n$ then a neighborhood of x is flat.

The main technical point of the proof of key lemma is the following:

3.4. Lemma. *Assume $x \in \mathcal{A}$ does not admit a conic neighborhood. Then there is a point $x' \in \mathcal{A}$ such that a neighborhood of x' is homothetical to a neighborhood of x and $\#(x') > \#(x)$.*

Proof. in next subsection $\square$

Proof of Key Lemma (3.2). Assume contrary; i.e., there is a point $x \in \mathcal{A}$ which does not admit a conic neighborhood.

Applying 3.4 to $x_0 = x$, we get a point x_1 with a neighborhood homothetic to a neighborhood of x_0 and $\#(x_1) \geqslant \#(x_0) + 1$. In particular, x_1 does not admit a conic neighborhood.

Therefore we can apply 3.4 again and again to get a sequence $x_0, x_1, \ldots, x_{n+1}$ in $\mathcal{A}$ such that $\#(x_{n+1}) \geqslant n + 1$. We arrive to a contradiction since $\#(z) \leqslant n$ for any $z \in \mathcal{A}$ $\square$

3A Proof of lemma 3.4

The next lemma shows that if some point x is on the same distance from finite set of others and correspondent distance square functions are 1-affine and x is not on the vertex line we can push x from this points, remaining on the same distance from them. All distances change as in Euclidean case.

3.5. Moving Lemma. *Let point $x \in A$ doesn't admit cone neighborhood and points $p_1, \dots, p_k \in A$, $r > 0$. Suppose that following conditions hold:*

(i) functions $f_1 = \frac{1}{2} \cdot \operatorname{dist}^2_{p_1}, \dots, f_k = \frac{1}{2} \cdot \operatorname{dist}^2_{p_k}$ are 1-affine in neighborhood $B_r(x)$

(ii) $|p_1 x| = \dots = |p_k x|$.

Then then there is a unique unit vector $v \in Span(\nabla f_1, \dots, \nabla f_k)$ so that $\angle(v, \nabla f_1) = \dots = \angle(v, \nabla f_k) = \alpha < \pi/2$ and a shortest path $\gamma : [0, r/4] \to A$, with $\gamma(0) = x$ and $\gamma'(0) = v$, so that for any $t \in [0, r/4]$ and $y = \gamma(t)$ the following holds:

1. *Some small neighborhoods of x and y are homothetic*
2. *$f_i(\gamma(t)) = |\nabla f_i| \cos(\alpha) t + \frac{1}{2} t^2$, in particular $|p_1 y| = \dots = |p_k y| > |p_1 x|$*
3. *$\angle(\gamma'(t)), \nabla f_1) = \dots = \angle(\gamma'(t)), \nabla f_k) < \alpha$ and $\#(F, y) = \#(F, x)$*
4. *suppose that for some $p \in A$ the function $f = \frac{1}{2} \cdot \operatorname{dist}^2_p$ is 1-affine in some neighborhood of y, $f(y) = f_i(y)$ and $\angle(\nabla_y f, \gamma'(t)) \neq \angle(\nabla_y f_i, \gamma'(t))$, then $\#(\{f, f_1, \dots, f_k\}, y) = \#(\{f_1, \dots, f_k\}, x) + 1$.*

proof

Since x doesn't have cone neighborhood we apply 2.9 and obtain that

$$\#_L(\nabla\{f_1, \dots, f_k\}, x) = \#_A(\nabla\{f_1, \dots, f_k\}, x) + 1$$

hence there is a unique unit vector $v \in Span(\nabla f_1, \dots, \nabla f_k)$ so that $\angle(v, \nabla f_1) = \dots = \angle(v, \nabla f_k) = \alpha < \pi/2$. We also obtain decomposition of $B_{r/4}(p)$ as subset of $\mathbb{R}^m \times K$, where K is a cone and $m = dim(Span(\nabla f_1, \dots, \nabla f_k)) - 1$. Then existence of a shortest path $\gamma : [0, r/4] \to A$, with $\gamma(0) = x$, $\gamma'(0) = v$ and properties 1-3 follows directly from this decomposition and 1-affinity of functions. To check 4 we note that equalities $|\nabla_y f_1| = \dots = |\nabla_y f_k|$ and $\angle(\gamma'(t)), \nabla f_1) = \dots = \angle(\gamma'(t)), \nabla f_k)$ means that endpoints of vectors $\nabla_y f_1, \dots, \nabla_y f_k$ in the space $Span(\nabla f_1, \dots, \nabla f_k)$ belong to intersection of a hyperplane with normal vector $\gamma'(t)$ with a sphere. Hence for any vector v with endpoint in affine hull of endpoints of $\nabla_y f_1, \dots, \nabla_y f_k$ and so that $|v| = |\nabla_y f_1|$ we would have $\angle(\gamma'(t)), \nabla f_1) = \angle(\gamma'(t)), v)$. Now since $|\nabla_y f| = |\nabla_y f_1|$ and $\angle(\nabla_y f, \gamma'(t)) \neq \angle(\nabla_y f_i, \gamma'(t))$ we have that

$$\#_A(\nabla\{f_1, \dots, f_k, f\}, x) = \#_A(\nabla\{f_1, \dots, f_k\}, x) + 1.$$

□

Proof of Lemma 3.4. For each i the sets V_i and $\mathfrak{C}_i$ are closed and their intersection is empty. Hence from Proposition 3.3, there exists $r_0 > 0$ so that for every i and $x \in V_i$ function $\frac{1}{2} \operatorname{dist}^2_{e_i}$ is 1-affine in $B_{4r_0}(x)$.

Now we fix $x \in A$ and suppose x doesn't have cone neighborhood. Applying Lemma 3.5 for x and $\{f_i | i \in J_x\}$, we can move x equidistantly from points e_i for $i \in J_x$ so it still lies in all V_i for $i \in J_x$ till it meets a domain V_j for some $j \notin J_x$.

Now more formally. Let $\gamma_0 : [0, r_0] \to A$ be the shortest path obtained in Lemma 3.5 $\gamma_0 : [0, r_0] \to A$. We have a dihotomia:

1. There is minimal value $t_0 > 0$, such that $\gamma_0(t) \in V_j$ for some $j_0 \notin J_x$. Set $y = \gamma_0(t_0) \in V_{j_0}$.
   In this case we can apply Moving Lemma 3.5 (4) with $p := e_{j_0}$, $f = \frac{1}{2}\operatorname{dist}^2_{e_{j_0}}$ (here we have angle inequality $\angle(\nabla_y f, \gamma'(t)) > \angle(\nabla_y f_i, \gamma'(t))$ since otherwise t_0 wouldn't be minimal). Then some small neighborhoods of x and y are homothetic and
   $$\begin{aligned}\#(y) \geqslant \#(\{f_i|i \in J_x\} \cup \{f_{j_0}\}, y) = \\ = \#(\{f_i|i \in J_x\}, x) + 1 = \\ = \#(x) + 1.\end{aligned}$$
2. The shortest path $\gamma_0([0, r_0])$ does not contain points in any V_j for $j \notin J_x$. In this case we apply Moving Lemma (3.5) recursively for $x_1 = \gamma_0(r_0)$ e.t.c. After k iteration we will have an estimate $f_i(x_k) > (|\nabla_x f_i| \cos(\alpha_0) r_0) \cdot k$, $i \in J_x$ where $\alpha_0 = \angle(\nabla_x f_i, \gamma_0'(0))$. Since diameter of A is finite this means again that after finite step we come to the case 1. □

4 Box is a flat orbifold

In this section we finish the proof of Main theorem I (1.2).

Note that according to Theorem 3.1 and Proposition 2.14, it is sufficient to show the following:

4.1. Theorem. *Let an n-dimensional polyhedral space $\mathcal{A}$ be a box. Then normal cone for each face of codimension 2 in $\mathcal{A}$ is isometric to one of the following spaces: $\mathbb{R}^2$, $\mathbb{R}_+ \times \mathbb{R}$, $\mathbb{R}_+ \times \mathbb{R}_+$ or a cone over a circle of length π.*

Let $\mathcal{A}$ be an n-box. We keep the same notation as before: e_i denote extremal points of $\mathcal{A}$, V_i corresponding Voronoi domain, $\mathfrak{C}_i$ the cut locus of e_i; $i \in \{1, 2, \ldots, 2^n\}$. A minimizing geodesic $[e_i e_j]$ between two extremal points will be called *edge*.

Let $p \in \mathcal{A}$ be a point which lies on face of codimension 2; i.e., $T_p = \mathbb{R}^{m-2} \times L$, where L denotes a 2-dimensional cone which does not have a line. Take all points in $\mathcal{A}$ with tangent cone isometric to p; its closure H will be called *hyperedge* (we name it this way since H has codimension 2 in $\mathcal{A}$).

4A Proof of theorem 4.1

4.2. Definition. *Let $\mathcal{A}$ be a box, $\triangle \subset \mathcal{A}$ be a flat simplex and e_i be a vertex. We say that $\triangle$ is* pressed down from e_i *if $\triangle \subset \mathfrak{C}_i$.*

Assume Δ is pressed down by e_j and $e_i \in \Delta$ be an other vertex. We say that Δ separates *e_i and e_j if there is a simplex $\Delta' \subset \Delta$ of the same dimension with vertex $e_i \in \Delta'$, so that*

$$int(\varphi_i^{-1}(\triangle')) \cap V_k = \emptyset \quad \textit{for every} \quad k \neq i, j.$$

We say that a hyperedge Q in $\mathcal{A}$ is pressed down by *e_j at $x \in Q$ if Q contains a k-dimensional simplex Δ with a vertex at x which is pressed down from e_j.*

We say that a hyperedge Q in $\mathcal{A}$ separates e_i *and* e_j *at* $x \in Q$ *if* Q *contains a* k*-dimensional simplex* Δ *which separates* e_i *and* e_j*.*

To prove Theorem 4.1, we need the following lemma:

4.3. Lemma. *Let $\mathcal{A}$ be a box and Q be a hyperedge. Then Q contains a flat $(n-2)$-simplex $\triangle$ which separates some pair of vertexes e_i and e_j.*

First let us show how Theorem 4.1 follows from Lemma 4.3.

Proof of Theorem 4.1. Let us introduce some notations:

- ⋄ K_i will denote the completion of $\mathcal{A} \setminus \mathfrak{C}_i$ equipped with intrinsic metric.
- ⋄ Clearly K_i is isometric to $2{\cdot}V_i$. Denote by $\psi_i : g_i^{-1}(V_i) \to K_i$ the homothety with center e_i and coefficient 2.
- ⋄ $g_i : K_i \to \mathcal{A}$ be the correspondent gluing map (that is piecewise linear).

Note that in these notations we have $g_i \circ \psi_i \circ g_i^{-1} = \varphi_i$.

We can assume $\triangle$ be sufficiently small so that $g_i^{-1}(int(\triangle))$ are disjoint isometric copies of $int(\triangle)$, let denote closers of these preimages: $\triangle_1, \dots, \triangle_l$ and correspondent preimages of e_j as: $e_j^1 \in \triangle_1, \dots, e_j^l \in \triangle_l$.

4.4. Claim. *In our conditions let point $x \in g^{-1}(int(\triangle)) \subset \partial K_i$. There are 2 possibilities:*

(1) if $\psi_i^{-1}(x) \notin \partial K_i$ then $T_x K_i = \mathbb{R}^{n-1} \times \mathbb{R}_+$

(2) if $\psi_i^{-1}(x) \in \partial K_i$ then $T_x K_i = \mathbb{R}^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$.

Proof. For every $y = \psi_i^{-1}(x)$ we know, that $T_y K_i$ contains isometric copy of $\mathbb{R}^{n-2} \times \mathbb{R}$. Hence $T_y K_i = \mathbb{R}^n$ or $T_y K_i = \mathbb{R}^{n-1} \times \mathbb{R}_+$. We know also that $\psi_i^{-1}(\triangle_1)$ is a flat $(n-2)$ simplex equidistant from e_i and e_j^1 with midpoint $\psi_i^{-1}(e_j^1)$ as a vertex. In small neighborhood U of y we will have that

$$g_i^{-1}(V_i) \cap U = \left\{ \, z \in U \,\middle|\, |ze_i| \leqslant |ze_j^1| \, \right\}.$$

Then distance functions from e_i and e_j^1 will divide U and $T_y K_i$ so that we have:

- ⋄ $T_x(g_i^{-1}(V_i)) = \mathbb{R}^{n-1} \times \mathbb{R}_+$ if $T_y K_i = \mathbb{R}^n$
- ⋄ $T_y(g_i^{-1}(V_i)) = \mathbb{R}^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$ if $T_y K_i = \mathbb{R}^{n-1} \times \mathbb{R}_+$. □

4.5. Claim. *For a point $x \in g^{-1}(\mathfrak{C}_i) \subset \partial K_i$ the condition $\psi_i^{-1}(x) \in \partial K_i$ implies $g_i(x) \in \partial\mathcal{A}$.*

Proof. Almost obvious because our space is polyhedral and $g_i(\partial K_i) \setminus \mathfrak{C}_i \subset \partial\mathcal{A}$. □

We can regard the space K_i as cutting of polyhedral $\mathcal{A}$ along $(n-1)$-polyhedral subspace $\mathfrak{C}_i$. The map g glues $\mathcal{A}$ back from K_i. Then if the point $x \in \mathfrak{C}_i$ has l preimages under g_i: $x_1, \dots, x_l \in K_i$ its tangent space T_x can be glued out of the tangent spaces $T_{x_1}, \dots, T_{x_l}$, we write this:

$$T_x = T_{x_1} \sqcup \dots \sqcup T_{x_l},$$

the gluing maps are $d_{x_1} g_i : T_{x_1} \to T_x, \dots, d_{x_l} g_i : T_{x_l} \to T_x$.

Fix x let $g_i^{-1}(x) = \{x_1, \dots, x_l\} \subset K_i$. Then we might have the following cases:

1. $\triangle \subset \partial\mathcal{A}$

(a) for some $1 \leqslant k_0 \leqslant l$ the point $\psi^{-1}(x_{k_0}) \notin \partial K_i$. Then by Claim 4.4 $T_{x_k} = \mathbb{R}^{n-1} \times \mathbb{R}_+$, $l = 1$ and $T_x = \mathbb{R}^{n-1} \times \mathbb{R}_+$.

(b) for all $k \in \{1, \ldots, l\}$ points $\psi^{-1}(x_k) \in \partial K_i$. Then $T_{x_k} = \mathbb{R}^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$ This only possible if $l = 1$ and $T_{x_k} = \mathbb{R}^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$ or $l = 2$ and $T_x\mathcal{A} = \mathbb{R}^{n-1} \times \mathbb{R}_+$.

2. $int(\triangle) \cap \partial\mathcal{A} = \emptyset$, in this case Claim 4.4 implies that for all $k \in \{1, \ldots, l\}$ points $\psi^{-1}(x_k) \notin \partial K_i$ and by lemma $T_{x_k} = \mathbb{R}^{n-1} \times \mathbb{R}_+$. This only possible if if $l = 1$ and $T_x\mathcal{A} = \mathbb{R}^{m-2} \times L$, where L is a cone over S^1 of length π or $l = 2$ and $T_x\mathcal{A} = \mathbb{R}^n$. $\square$

Proof of Lemma 4.3. It is sufficient to prove the following two claims.

4.6. Claim. *Let $\mathcal{A}$ be a box then for any face of codimension 2 is pressed down from some vertex e_i.*

Proof of the claim. Suppose there exists at least one vertex $e_i \notin Q$, then Q is pressed down from e_i.

Otherwise consider any flat n-simplex with vertexes in $\{e_1, \ldots, e_{2^n}\}$ say $\triangle_{e_{i_0}, \ldots, e_{i_n}}$. The existence of such a simplex can be proved by using construction as in the proof of 3A: moving out from vertexes we can find a point $x \in \mathcal{A}$ with $\#(x) = n$ and from 2.1 follows that correspondent $n + 1$ vertexes form flat n simplex. Since codimension of Q is 2, Q has to be pressed down at e_{i_0} from one of the remaining vertexes $e_{i_1}, e_{i_2}, \ldots, e_{i_n}$. $\square$

4.7. Key claim. *Let e_i and e_j be two vertexes and Q be a hyperedge in a box $\mathcal{A}$ and $e_j \in Q$. Assume e_i presses down Q at e_j but Q does not separate e_i and e_j. Then there is $k \neq i, j$ so that*

$$\max\{|e_k e_i|, |e_k e_j|\} < |e_i e_j|$$

and one of the following holds:

◇ *e_k press down face Q at e_j.*

◇ *$e_k \in Q$ and e_i presses down Q at e_k.*

To prove the Key Claim 4.7 we will need the following lemma:

4.8. Lemma. *For any vertexes e_i, e_k and a point $x \in V_i \cap V_k$ there is a shortest path $[\varphi_i(x)e_k]$ inside $\mathfrak{C}_i$.*

Proof. By Lemma 1.8 there is a flat triangle $e_i e_k \varphi_i(x)$ with median $[xe_k]$ and right angle in e_k. If some point of the edge $[\varphi_i(x)e_k]$ of this triangle would not be in $\mathfrak{C}_i$ then we would have $\operatorname{diam} \Sigma_{e_k} > \pi/2$, contradiction. $\square$

Proof of Key Claim 4.7. In conditions of our claim there is $(n-2)$-simplex Δ with vertex $m \in \varphi_i^{-1}(e_j)$ so that $\varphi_i(\Delta) \subset Q$ and $\Delta \subset V_i \cap V_k$ for some $k \neq i, j$. Then by Lemma 1.8 there is a flat triangle $e_i e_j e_k$ with median $[e_k m]$ and right angle in e_k. Then

$$\max\{|e_k e_i|, |e_k e_j|\} < |e_i e_j|.$$

Now if e_k press down face Q at e_j proof finished. Suppose contrary, we can assume that $int\varphi_i(\Delta) \subset \mathcal{A} \setminus \mathfrak{C}_k$. By Lemma 4.8 for every point $y \in \varphi_i(\Delta)$ there is a shortest path $[ye_k]$ inside $\mathfrak{C}_i$, if in addition $y \notin \mathfrak{C}_k$ then $[ye_k] \subset Q$. Then points of all such shortest paths for $y \in int\varphi_i(\Delta)$ form $(n-2)$-dimensional subset of Q. In particular $e_k \in Q$ and e_j presses down Q at e_k. $\square$

5 The structure of the action

In the last part of the paper we prove Main theorem II.

Assume $\Gamma \curvearrowright \mathbb{R}^n$ is a descrete isometric action and $\pi : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ denotes projection. Let us denote by $\mathcal{E}$ the set of isolated fixed points of some subgroups of Γ. Clearly $\mathbb{R}^n/\Gamma$ is a box iff the number of Γ-orbits in $\mathcal{E}$ is 2^n.

For any $x \in \mathcal{E}$ we will denote by V_x its Voronoy cell; i.e.

$$V_x = \{\ z \in \mathbb{R}^n \mid |z - x| \leqslant |z - y| \text{ for any } y \in \mathcal{E}\ \}.$$

Given $x \in \mathbb{R}^n$, let us denote by $\Gamma_x^\# \subset O(n)$ the rotational part of stabilizer of x in Γ.

If for any $x, y \in \mathcal{E}$ the condition $\dim(V_x \cap V_y) = n - 1$ implies $\Gamma_x^\#(x - y) = \{x - y, y - x\}$ then we say that $\Gamma \curvearrowright \mathbb{R}^n$ has *reflection property.* Further, if for any $X, Y \in \mathcal{E}$ the condition $\dim(V_x \cap V_y) = n - 1$ implies that the midpoint $(x + y)/2$ lies in the interior of the face $V_x \cap V_y$ then we say that $\Gamma \curvearrowright \mathbb{R}^n$ has *midpoint property.*

Using above terminology Main theorem II can be reformulated in the following statement:

5.1. Claim. *Let the number of Γ-orbits in $\mathcal{E}$ be 2^n. Then the set $\mathcal{E} \subset \mathbb{R}^n$ is a lattice; i.e. there is an ordered basis $(\alpha_1, \alpha_2, \dots \alpha_n)$ of $\mathbb{R}^n$ such that for any $x, y \in \mathcal{E}$, we have*

$$y - x = k_1{\cdot}\alpha_1 + k_2{\cdot}\alpha_2 + \dots + k_n{\cdot}\alpha_n$$

for some integers $k_1, k_2, \dots k_n$.

Moreover $\pi(x) = \pi(y)$ if and only if all $k_1, k_2, \dots k_n$ above are even.

We can reduce this claim to the following two statements:

5.2. Proposition. *Assume $\Gamma \curvearrowright \mathbb{R}^n$ is an action for a box. Then $\Gamma \curvearrowright \mathbb{R}^n$ has reflection and midpoint properties.*

Proof Proof in section 6.

5.3. Proposition. *Let Γ be a discrete cocompact subgroup of isometries of $\mathbb{R}^n$. Let $\mathcal{E}$ be the nonempty set of singular points for Γ and $\mathcal{E}$ has reflection and midpoint property. Then there is an n-generating set $a_1, \dots, a_n$ for $\mathcal{E}$ so that for any $X \in \mathcal{E}$ we have $\Gamma_X^\#(a_i) = \{a_i, -a_i\}$.*

Proof Proof is in section 7.

6 Properties of a group action for a box.

In this section we prove Proposition 5.2. Here $\mathcal{A}$ is a box and $\Gamma \curvearrowright \mathbb{R}^n$ the correspondent group action. We describe properties of a group action via geometry of a factor space and Proposition 5.2 is direct consequence of Claim 6.1 and Claim 6.5 below.

In case if edge $[e_i e_j]$ of a box $\mathcal{A}$ does not intersect any Voronoij domain except V_i and V_j, we say that $[e_i e_j]$ is a *simple edge.*

The first aim of this section is to prove the following:

6.1. Claim. *For any $1 \leqslant i < j \leqslant 2^n$ we have: $dim(V_i \cap V_j) = n-1$ iff edge e_ie_j is simple. Moreover, in this case a shortest path between e_i and e_j is unique.*

We give firstly one characterizing property of simple edge.

6.2. Lemma. *Let M be midpoint of an edge e_ie_j and unit vector $v \in T_M\mathcal{A}$ be direction from M to e_i. The following properties are equivalent*

(1) edge e_ie_j is simple

(2) for every point in unit tangent space $x \in \Sigma_M V_i$ there is a shortest path in $\Sigma_M V_i$ of length $\pi/2$ with initial point v containing x.

proof Let $\Sigma_{ij}^{\perp}$ be subset of tangent space $\Sigma_{M_{ij}}\mathcal{A}$ containing all vectors perpendicular to the edge e_ie_j. Then it is not difficult to see that both properties (1) and (2) are equivalent to the following:

(3) The set $\Sigma_{ij}^{\perp} = \Sigma_{M_{ij}} V_i \cap \Sigma_{M_{ij}} V_j$ and tangent space of V_i naturally splits as: $\Sigma_{M_{ij}} V_i = \Sigma_{ij}^{\perp} \times \mathbb{R}^+$.

□

6.3. Lemma. *Let S^k be a k-dimensional sphere, Γ discrete subgroup of isometries of S^k, $B = S^k/\Gamma$ with projection $p : S^k \to B$ and $diamB \leqslant \pi/2$. Let point $e \in S^k$, we denote the equator $S^{\perp} = \{y \in S^k | |ye| = \pi/2\}$. Let for some $(k-1)$-dimensional set $F \subset B$ so that $|p(e)x| = \pi/2$ for all $x \in F$. Then we will have*

(1) for every point $x \in B$ there is a shortest path of length $\pi/2$ with initial point $p(e)$ containing x or equivalently for orbit $\Gamma(e) = \{e, e^-\}$, where e^- is the opposite point for e.

(2) Let in addition we have Voronoy decomposition in B with respect to some finite set $\{e_1, \dots, e_l\} \subset B$ where $e_1 = p(e)$, suppose that $F \subset \mathfrak{V}_{e_1}(B, e_1, \dots, e_l)$ Then Voronoy set is unique i.e. $l = 1$.

Proof. (1) We regard Voronoj decomposition of $\mathbb{S}^k$ with respect the set of points $\Gamma(e)$. Let denote $S^{\perp} = \left\{ x \in S^k \,\middle|\, |xe| = \frac{\pi}{2} \right\}$ and let $F' = p^{-1}(F) \cap S^{\perp}$ then $\dim F' = k-1$ and (see 2.15 (2)) $F' \subset \partial\mathfrak{V}_e(\mathbb{S}^k, \Gamma(e))$. Hence some $(k-1)$-dimensional subset of F' is subset of the boundary of the other Voronoi set, obviously the only possible place for the vertex of this Voronoi set is e^-. Then $\Gamma(F') = p^{-1}(F) \subset S^{\perp}$, otherwise we would obtain $dist(p(e), F) < \frac{\pi}{2}$. Since $\dim(p^{-1}(F)) = k-1$ we have $\Gamma(e) = \{e, e^-\}$. Hence (1).

(2) Suppose contrary, then for point e_2 we will have $|e_2x| \geqslant |e_1x| = \pi/2$ for any $x \in F$, hence diameter bound implies $|e_2x| = \pi/2$ for any $x \in F$. F is $k-1$-dimensional and we can find $e_2^* \in p^{-1}(e_2)$ and $k-1$-dimensional $F^* \subset p^{-1}(F)$ so that $|e_2^*x| = \pi/2$ for any $x \in F^*$. Hence $e_2^* \in \{e, e^-\}$, contradiction. □

Proof of claim 6.1 Set $B = \Sigma_{e_j}\mathcal{A}$, let $s_1 = s, s_2, \dots, s_l$ be all shortest paths from e_j to e_i, set $a_i = s_i'(0) \in B$. We regard Voronoj decomposition $\mathfrak{V}_{a_i}(B, a_1, \dots, a_l)$ and prove the following:

6.4. Lemma. *Suppose $dim(V_i \cap V_j) = n-1$. In above notations for one of the shortest paths from e_j to e_i, say s_1 there is an $(n-2)$-dimensional set $F^* \subset \mathfrak{V}_{a_1}(B, a_1, \dots, a_l) \subset B$ so that $|a_1x| = \pi/2$ for any $x \in F^*$.*

Proof. Lemma 1.8 implies that there is continuous map that assigns to every point $x \in V_i \cap V_j$ the midpoint $M_{ij}(x)$ of some shortest paths between e_i and e_j

and the unique shortest path between x and $M_{ij}(x)$ is subset of $V_i \cap V_j$. Hence for every connected set $L \subset V_i \cap V_j$ there is one and the same correspondent midpoint $M_{ij}(K)$ and cone over L with center $M_{ij}(K)$ is subset of $V_i \cap V_j$. Then we can find $(n-1)$-dimensional flat simplicial subset $\Delta \subset V_i \cap V_j$ with vertex $M_{ij}(\Delta)$ - midpoint of some shortest path $s : [0, |e_i e_j|] \to \mathcal{A}$ from e_j to e_i. Then we can take $s_1 = s$ and $F^* = dg_i \psi_i|_{M_{ij}(\Delta)}(\Sigma_{M_{ij}} \Delta) \subset \Sigma_{e_j} \mathcal{A} = B$. It follows from Lemma 2.13 that $F^* \subset \mathfrak{V}_{a_1}(B, a_1, \ldots, a_l)$. □

We know that $B = S^{n-1}/\Gamma^{\#}_{e_j}$ and Lemma 6.4 implies that conditions of Lemma 6.3 are fulfilled for $B = \Sigma_{e_j} \mathcal{A}$, $p(e) = s'(0)$ and $F = F^*$. Now from Lemma 6.3 (2) we obtain that shortest path s_1 between e_i and e_j is unique and from (1) for every point $x \in \Sigma_{e_j} \mathcal{A}$ there is a shortest path of length $\pi/2$ with initial point $s'_1(0)$ containing x. It follows that $g_i^{-1}(e_j)$ is a unique point, say e_j^*. We note that $\Sigma_{e_j^*} K_i$ is isometric to the completion of $B \setminus \mathfrak{C}_{s'_1(0)}$ with intrinsic metric, where $C_{s'_1(0)}$ is cutlocus of $s'_1(0)$ in B. Combining conclusion(2) of Lemma 6.3 with Let s_*, uniquely defined liftings of s_1 under g_i. Then for every point $x \in \Sigma_{e_j^*} K_i$ there is a shortest path of length $\pi/2$ with initial point $s'_*(0)$ containing x. Since ψ_i is local isometry we have that condition (2) of Lemma 6.2 holds and the claim follows.

□

In next Claim we formulate properties of simple edge that we need to prove reflection and midpoint properties.

6.5. Claim. *Let $e_i e_j$ be simple edge. Then*

(1) Let $e'_i, e'_j \in \mathbb{R}^n$ be preimages of e_i, e_j so that $|e'_i e'_j| = |e_i e_j|$, then $\Gamma_{e_i}(e_j) = \{e_j, e_i + e_j e_i\}$.

(2) Let $m_{ij} = (e'_i + e'_j)/2$ then m_{ij} is an intrinsic point of $(n-1)$-intersection $V_{e'_i} \cap V_{e'_j}$

Proof. *To prove (1) we can as earlier apply Lemma 6.3 . It follows from definition of simple edge that $m_{ij} \in V_{e'_i} \cap V_{e'_j}$ and $m_{ij} \notin V_e$, for any $e \in \mathcal{E} \setminus \{e'_i, e'_j\}$, hence (2).*

□

7 Reflection and midpoint property gives a lattice of fixed points.

In this section we prove Proposition 5.3.

We note firstly that it is sufficient to find any finite generating set for $\mathcal{E}$ with the same property:

7.1. Lemma. *Suppose $S \in \mathbb{R}^n$ is a lattis with generating set $a_1, \ldots, a_s$ and origin O, G is a subgroup of isometries of $\mathbb{R}^n$ with the property $G^{\#}_O(a_i) = \{a_i, -a_i\}$. Then there is an n-generating $b_1, \ldots, b_n$ for S with the same property $G^{\#}_O(b_i) = \{b_i, -b_i\}$.*

Proof. We define equivalence relation on generating set $a_1, \ldots, a_s$. We set $a_{i^*} \sim a_{j^*}$ if we can find sequence $a_{i^*} = a_{i_1}, \ldots, a_{i_k} = a_{j^*}$ so that $\angle(a_{i_m}, a_{i_{m+1}}) \neq \pi/2$. Let denote $q_1, \ldots, q_l$ equivalence classes.

Let regard lattices $S^i = \{O + \sum_{a \in q_i} n_a a, n_a \in \mathbb{Z}\}$. Then

$$S = \bigoplus_{i=1}^{l} \mathcal{S}^i.$$

Let note that for any $X, Y \in S^i$

$$G_X(XY) = \{XY, -XY\}.$$

Let $d(i)$ be dimension of affine hull of S^i. Then for every lattice S^i we can find points $X_1^i, \dots, X_{d(i)}^i \in S^i$ so that $OX_1^i, \dots, OX_{d(i)}^i$ is basis for S^i.

Then the set $B = \cup_{i=1}^{l}\{OX_1^i, \dots, OX_{d(i)}^i\}\}$ is n-basis for E with the property that for every $a \in B$, $X \in \mathcal{E}$

$$\Gamma_X^{\#}(a) = \{a, -a\}.$$

□

For $X \in \mathcal{E}$ we will denote

$$\mathfrak{S}(X) = \{Y \in \mathcal{E} | dim(V_X \cap V_Y) = n - 1\}.$$

Then by above the proposition 5.3 will follow from the following claim:

7.2. Claim. *For every $X \in \mathcal{E}$ the set $\mathcal{E}$ is a lattis with origin X and generating vectors $\{\overrightarrow{XY}\}_{Y \in \mathfrak{S}(X)}$.*

The main technical point of the proof is the following:

7.3. Lemma. *Let $X \in \mathcal{E}$ $Y, Z \in \mathfrak{S}(X)$, let denote $Z^* = Y + \overrightarrow{XZ}$. then*

$$Z^* \in \mathcal{E}.$$

proof the proof is in subsection 7B

proof that Lemma 7.3 implies Claim 7.2

For any points $X, Y \in \mathbb{R}^n$ and midpoint M_{XY} between them let denote the halfspace

$$H(X, Y) = \{Z \in \mathbb{R}^n | \overrightarrow{M_{XY}Z} \cdot \overrightarrow{XY} \leqslant 0\},$$

then obviously

$$V_X = \cap_{Y \in \mathfrak{S}(X)} H(X, Y) = \cap_{Y \in \mathcal{E} \setminus X} H(X, Y) \qquad ❺.$$

We show firstly that for every $X, Y \in \mathcal{E}$

$$\mathfrak{S}(Y) = \mathfrak{S}(X) + \overrightarrow{XY} \qquad ❻.$$

Indeed lemma 7.3 implies that for any $X, Y \in \mathcal{E}$ so that $Y \in \mathfrak{S}(X)$ we have $\mathfrak{S}(X) + \overrightarrow{XY} \subset \mathcal{E}$ and hence by ❺ $V_Y \subset V_X + \overrightarrow{XY}$. Then changing X and Y we obtain equality ❻ in this case. The general case for any $X, Y \in \mathcal{E}$ can be obtained by joining X and Y with a chain of $\mathfrak{S}$-edges.

To show Claim 7.2 it is sufficient to prove that for any $X, Y, Z \in \mathcal{E}$ we have $X + \overrightarrow{XY} + \overrightarrow{XZ} \in \mathcal{E}$ $X - \overrightarrow{XY} \in \mathcal{E}$. The second inclusion needs the central symmetry of the set $\mathfrak{S}(X)$, that follows from the reflection property. After this both inclusions can be proved by using ❻ and joining correspondent points with a chain of $\mathfrak{S}$-edge. □

In the next subsection we prove some technical facts for lemma 7.3.

7A Some technical facts about group action with reflection and midpoint properties.

7.4. Lemma. *For any $X \in \mathcal{E}$ the set $\mathfrak{S}(X)$ has at at least n points $X_1, \dots, X_n$ so that vectors $\overrightarrow{XX_1}, \dots, \overrightarrow{XX_n}$ are linearly independent.*

Proof. Follows from first equality in ❺ since V_X is compact. □

We denote stabilizer of a line: $\Gamma_{X,Y} = \Gamma_X \cap \Gamma_Y$.

7.5. Sublemma. *Let $X \in \mathfrak{S}(Y)$. Then for every v, so that $\Gamma_{X,Y}(v) = \{v\}$ we have $\Gamma_Y(v) = \{v, -v\}$.*

Proof. We can find $X_1, \dots, X_{n-1} \in \mathfrak{S}(Y)$ so that vectors $\overrightarrow{YX}, \overrightarrow{YX_1}, \dots, \overrightarrow{YX_{n-1}}$ be linearly independent. Let $P = \{v \in \mathbb{R}^n | \Gamma^{\#}_{X,Y}(v) = v\}$. Changing order if necessary we can assume that $X = X_0, X_1, \dots, X_k \in P$ and $X_{k+1}, \dots, X_{n-1} \notin P$. We know that $\Gamma_Y(\overrightarrow{YX_i}) = \{\overrightarrow{YX_i}, -\overrightarrow{YX_i}\}$ for every $i = 0, \dots, n-1$. Then regarding group for decomposition in $v = v_0\overrightarrow{YX_0} + \dots + v_{n-1}\overrightarrow{YX_{n-1}}$ we easily obtain that for every $v \in P$ we have $v_{k+1} = \dots = v_{n-1} = 0$, i.e. $P = Span(X_0, \dots, X_k)$. Then for every $v \in P$ we have $\Gamma_Y(v) = \{v, -v\}$. Sublemma follows. □

7.6. Notation. *For any point $X \in \mathbb{R}^k$ we will denote*

$$\mathfrak{c}_X : \mathbb{R}^k \to \mathbb{R}^k$$

central symmetry with center X.

For points $X_1, \dots, X_l$ we denote $\langle X_1, \dots, X_l \rangle$ the affine hull of this points.

7.7. Lemma. *Let points $X, Y, Z \in \mathcal{E}$ and $Y, Z \in \mathfrak{S}(X)$. Let denote point $Z^* = Y + \overrightarrow{XZ}$. Suppose that $\Gamma_{X,Y} = \Gamma_{X,Z}$ or equivalently for affine hull $\langle X, Y, Z \rangle$ we have $\Gamma_X|_{\langle X,Y,Z \rangle} = \{id|_{\langle X,Y,Z \rangle}, \mathfrak{c}_X|_{\langle X,Y,Z \rangle}\}$.*

Then

1) $\Gamma_Y|_{\langle X,Y,Z \rangle} = \{id|_{\langle X,Y,Z \rangle}, \mathfrak{c}_Y|_{\langle X,Y,Z \rangle}\}$

2) $Z^* \in \mathcal{E}$

Proof. 1) Condition of lemma implies that $\Gamma_{X,Y}(\overrightarrow{XZ}) = \{\overrightarrow{XZ}\}$ and $\overrightarrow{XZ} = \overrightarrow{YZ^*}$ then by sublemma $\Gamma_Y(\overrightarrow{YZ^*}) = \{\overrightarrow{YZ^*}, -\overrightarrow{YZ^*}\}$. Then for every $\gamma \in \Gamma_Y$ we have $\gamma(\overrightarrow{XZ}) = \overrightarrow{XZ}$ and $\gamma(\overrightarrow{YZ^*}) = \overrightarrow{YZ^*}$ or $\gamma(\overrightarrow{XZ}) = -\overrightarrow{XZ}$ and $\gamma(\overrightarrow{YZ^*}) = -\overrightarrow{YZ^*}$. 1) follows.

2) For every $v \neq 0$ we want to find $\gamma \in \Gamma_{Z^*}$ so that $\gamma(v) \neq v$. We regard two possibilities.

Firstly if $\Gamma_{X,Y,Z}(v) \backslash \{v\} \neq \emptyset$ we can find required element $\gamma \in \Gamma_{X,Y,Z} \subset \Gamma_{Z^*}$.

Secondly let $\Gamma_{X,Y,Z}(v) = \{v\}$ then for arbitrary three elements $\gamma_X \in \Gamma_X \setminus \Gamma_{X,Y,Z}$, $\gamma_Y \in \Gamma_Y \setminus \Gamma_{X,Y,Z}$, $\gamma_Z \in \Gamma_Z \setminus \Gamma_{X,Y,Z}$ by sublemma we will have $\gamma_X(v) = -v, \gamma_Z(v) = -v, \gamma_Z(v) = -v$. Then $\gamma_X \circ \gamma_Y \circ \gamma_Z(v) = -v$ and $\gamma_X \circ \gamma_Y \circ \gamma_Z(Z^*) = Z^*$.

□

7.8. Lemma. *Let points $X, Y, Z \in \mathcal{E}$ and $Y, Z \in \mathfrak{S}(X)$. Let $\angle YXZ \neq \pi/2$. Then $\Gamma_X|_{\langle X,Y,Z \rangle} = \{id|_{\langle X,Y,Z \rangle}, \mathfrak{c}_X|_{\langle X,Y,Z \rangle}\}$ or equivalently $\Gamma_{X,Y} = \Gamma_{X,Z}$.*

Proof. Obvious since points Y, Z could be only reflected and isometry preserves angles. $\square$

We denote the common face

$$H^{XY} = V_X \cap V_Y$$

for $X, Y \in \mathcal{E}$.

7.9. Lemma. *Let $X \in \mathcal{E}$, $Y, Z, Q \in \mathfrak{S}(X)$ and $QX, ZX \perp YX$. Then the midpoint $(Y + X)/2$ is intrinsic point of H^{XQ}.*

Proof. Arguing by contradiction, follows from midpoint property. $\square$

7B Proof of Lemma 7.3

We regard two possibilities:

(1)$\Gamma_{X,Y} \neq \Gamma_{X,Z}$

in this case $XZ \perp XY$ and for every $Q \in \mathfrak{S}(X)$ we have $XQ \perp XY$ or $XQ \perp XZ$ because of lemma 7.8. Let denote $N = H^{XZ} \cap H^{XY} \cap V(X)$, firstly we show that $dim(N) = n-1$: suppose contrary then there would be $Q \in \mathfrak{S}(X)$, so that $(Z + Y)/2 \notin intH^{XQ}$, but this is impossible because of Lemma 7.9.

So N is $(n-1)$-face of V_Y, hence there is $Z^{**} \in \mathfrak{S}(Y)$ so that $N \subset \partial H^{Y,Z^{**}}$. Then there are two cases: $Z^{**}YX = \pi/2$ then $Z^* = Z^{**}$ and proof is finished or $Z^{**}YX \neq \pi/2$ then by Lemma 7.8 and then Lemma 7.7 $\Gamma_{Y,Z^{**}} = \Gamma_{X,Y} = \Gamma_{X,Z}$ - contradicts to initial assumption.

(2) $\Gamma_{X,Y} = \Gamma_{X,Z}$

then we are in condition of Lemma 7.7.

References

[1] Alexander, S., Bishop, R., *A cone splitting theorem for Alexandrov spaces* Pacific journal of mathematics, Vol. 218, N 1, 2005 , 1-16

[2] Alexander, S, Kapovitch, V., Petrunin, Alexandrov geometry

[3] Alexander, S, Kapovitch, V., Petrunin, A. *Alexandrov meets Kirszbraun* Proc. Gokova Geometry-Topology Conference 2010, S. Akbulut, D. Auroux, T. Onder, eds., International Press (2011), 88- 109.

[4] Yu. Burago, M. Gromov and G. Perel'man *A. D. Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk 47 (1992), no. 2(284), 3–51, 222, translation in Russian Math. Surveys 47 (1992), 1–58.

[5] Danzer, L., Grünbaun, B., *Über zwei Probleme bezglich konvexer Körper von P. Erdős und von V. L. Klee* Math. Z., 1962, Bd 79, S. 95–99.

[6] Erdős, P. *Some unsolved problems* Michigan Math. J., 1957, no. 4, p. 291–300.

[7] Grove, K.; Petersen, P. *A radius sphere theorem*, Invent. Math. 112 (1993), 577583.

[8] Lang Shroueder *Kirszbraun's theorem and metric spaces of bounded curvature*, Geom. Funct. Anal. (GAFA) 7 (1997), 535-560

[9] Lebedeva N. Number of subgroups in a Bieberbach group, mathoverflow.net/questions/13714.

[10] Perelman, *Elements of Morse theory on Aleksandrov spaces* (in Russian). St. Petersbg. Math. J. 5 (1): 205213.

[11] Perelman, G. Ya.; Petrunin, A. M. *Extremal subsets in Aleksandrov spaces and the generalized Liberman theorem.* (Russian) Algebra i Analiz 5 (1993), no. 1, 242–256; translation in St. Petersburg Math. J. 5 (1994), no. 1, 215–227

[12] Perelman, G.; Petrunin, A. Quasigeodesics *Quasigeodesics and gradient curves in Alexandrov spaces*, preprint, University of California at Berkeley, 1994.

[13] Petrunin, A., Parallel transportation for Alexandrov space with curvature bounded below. GAFA, Vol. 8 (1998) 123–148

[14] Petrunin, A., *Semiconcave functions in Alexandrov's geometry.* Surveys in Differential Geometry XI.